PRIFYSGOL glyndŵr UNIVERSITY

Glyndŵr University
Glyndŵr University Research Online

Aeronautical Engineering | Engineering

6-1-2012

Vibration and Buckling of Cross-Ply Composite Beams using Refined Shear Deformation Theory

Thuc Vo
Glyndwr University, t.vo@glyndwr.ac.uk

Fawad Inam
Glyndwr University, f.inam@glyndwr.ac.uk

Follow this and additional works at: http://epubs.glyndwr.ac.uk/aer_eng

Part of the Manufacturing Commons, Mechanics of Materials Commons, Polymer and Organic Materials Commons, Structural Materials Commons, and the Structures and Materials Commons

Recommended Citation
Vo., T & Inam, F. (2012) "Vibration and Buckling of Cross-Ply Composite Beams using Refined Shear Deformation Theory" [Paper presented to the 2nd International Conference on Advanced Composite Materials and Technologies for Aerospace Applications held at Glyndŵr University, 11-13th June, 2012]. Wrexham: Glyndŵr University.

This Conference Paper is brought to you for free and open access by the Engineering at Glyndŵr University Research Online. It has been accepted for inclusion in Aeronautical Engineering by an authorized administrator of Glyndŵr University Research Online. For more information, please contact d.jepson@glyndwr.ac.uk.

Vibration and Buckling of Cross-Ply Composite Beams using Refined Shear Deformation Theory

Abstract
Vibration and buckling analysis of cross-ply composite beams using refined shear deformation theory is presented. The theory accounts for the parabolical variation of shear strains through the depth of beam. Three governing equations of motion are derived from the Hamilton's principle. The resulting coupling is referred to as triply coupled vibration and buckling. A two-noded C1 beam element with five degree-of-freedom per node is developed to solve the problem. Numerical results are obtained for composite beams to investigate modulus ratio on the natural frequencies, critical buckling loads and load-frequency interaction curves.

Keywords
Composite beams, Refined shear deformation theory, Triply coupled vibration and buckling

Disciplines
Aerospace Engineering | Engineering | Manufacturing | Mechanical Engineering | Mechanics of Materials | Polymer and Organic Materials | Structural Materials | Structures and Materials

Comments
Copyright © 2012 Glyndŵr University and the authors, all rights reserved. This article was first presented at the **2nd International Conference on Advanced Composite Materials and Technologies for Aerospace Applications**, June 11-13, 2012, Wrexham, UK and published in the conference proceedings by Glyndŵr University.

Permission to copy, reprint/republish this material for advertising or promotional purposes or for creating new collective works for resale or redistribution must be obtained from Glyndŵr University. By choosing to view this document, you agree to all provisions of the copyright laws protecting it. It is published here with the permission of the Authors, and the full proceedings are available to purchase at http://www.lulu.com/shop/richard-day-and-sergey-reznik/advanced-composite-materials-and-technologies-for-aerospace-applications/paperback/product-20214156.html;jsessionid=566556BF08A9459FC8807B9BF9878A8D#productDetails

This conference paper is available at Glyndŵr University Research Online: http://epubs.glyndwr.ac.uk/aer_eng/7

Vibration and Buckling of Cross-Ply Composite Beams using Refined Shear Deformation Theory

Thuc Vo, Fawad Inam

School of Mechanical, Aeronautical and Electrical Engineering, Glyndŵr University, Plas Coch, Mold Road, Wrexham, LL11 2AW, UK

Abstract: Vibration and buckling analysis of cross-ply composite beams using refined shear deformation theory is presented. The theory accounts for the parabolical variation of shear strains through the depth of beam. Three governing equations of motion are derived from the Hamilton's principle. The resulting coupling is referred to as triply coupled vibration and buckling. A two-noded C^1 beam element with five degree-of-freedom per node is developed to solve the problem. Numerical results are obtained for composite beams to investigate modulus ratio on the natural frequencies, critical buckling loads and load-frequency interaction curves.

Key Words: Composite beams, Refined shear deformation theory, Triply coupled vibration and buckling.

1. Introduction

Structural components made with composite materials are increasingly being used in various engineering applications due to their attractive properties in strength, stiffness, and lightness. Understanding their dynamic and buckling behaviour is of increasing importance. The classical beam theory (CBT) known as Euler-Bernoulli beam theory is the simplest one and is applicable to slender beams only. For moderately deep beams, it overestimates buckling loads and natural frequencies due to ignoring the transverse shear effects. The first-order beam theory (FOBT) known as Timoshenko beam theory is proposed to overcome the limitations of the CBT by accounting for the transverse shear effects. Since the FOBT violates the zero shear stress conditions on the top and bottom surfaces of the beam, a shear correction factor is required to account for the discrepancy between the actual stress state and the assumed constant stress state. To remove the discrepancies in the CBT and FOBT, the higher-order beam theory (HOBT) is developed to avoid the use of shear correction factor and have a better prediction of response of laminated beams. The HOBTs can be developed based on the assumption of higher-order variations of in-plane displacement or both in-plane and transverse displacements through the depth of the beam. Many numerical techniques have been used to solve the dynamic and/or buckling analysis of composite beams using HOBTs. Some researchers studied the free vibration characteristics of composite beams by using finite element (Chandrashekhara and Bangera, 1992; Marur and Kant, 1996; Shi and Lam, 1999; Murthy et. al., 2005; Subramanian, 2006). Khdeir and Reddy (1996, 1997) developed analytical solutions for free vibration and buckling of cross-ply composite beams with arbitrary boundary conditions in conjunction with the state space approach. Analytical solutions were also derived by Kant et al. (1997, 2001) and Zhen and Wanji (2008) to study vibration and buckling of composite beams. By using the method of power series expansion of displacement components, Matsunaga (2001) analysed the natural frequencies and buckling stresses of composite beams. Aydogdu (2005, 2006a, 2006b) carried out the vibration and buckling analysis of cross-ply and angle-ply with different sets of boundary conditions by using Ritz method. Jun et al. (2009, 2011) introduced the dynamic stiffness matrix method to solve exactly the free vibration and buckling problems of axially loaded composite beams with arbitrary lay-ups. Although the HOBTs offer a slight improvement in accuracy compared to the FOBT, they are computationally more demanding due to higher-order terms included in the theory. Hence, there is a scope to develop accurate refined shear deformation theory which is simple to use.

In this paper, which is extended from previous research (Vo and Thai, 2012), vibration and buckling analysis of composite beams using refined shear deformation theory is presented. The displacement field of the present theory is chosen based on the following assumptions: (1) the axial and transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments; (2) the bending component of axial displacement is similar to that given by the CBT; and (3) the shear component of axial displacement gives rise to the higher-order variation of shear strain and hence to shear stress through the depth of the beam in such a way that shear stress vanishes on the top and bottom surfaces. The most interesting feature of this theory is that it satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factors. The three governing equations of motion are derived from the Hamilton's principle. The resulting coupling is referred to as triply coupled vibration and buckling. A two-noded C^1 beam element with five degree-of-freedom per node which accounts for shear deformation effects and all coupling coming from the material anisotropy is developed to solve the problem. Numerical results are obtained for

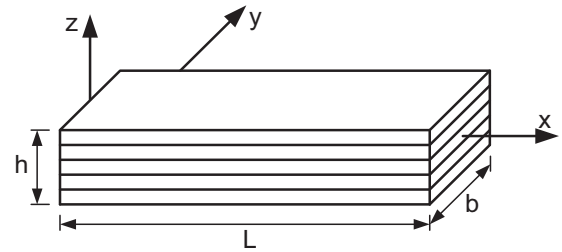


Figure 1. Geometry of a laminated composite beam.

composite beams to investigate effects of fiber orientation and modulus ratio on the natural frequencies, critical buckling loads and load-frequency interaction curves as well as corresponding mode shapes.

2. Kinematics

A laminated composite beam made of many plies of orthotropic materials in different orientations with respect to the x-axis, as shown in Fig. 1, is considered.

Based on the assumptions made in the preceding section, the displacement field of the present theory can be obtained as:

$$\begin{aligned} U(x,z,t) &= u(x,t) - z\frac{\partial w_b(x,t)}{\partial x} + \\ &+ z\left[\frac{1}{4} - \frac{5}{3}\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s(x,t)}{\partial x} \\ V(x,z,t) &= 0 \\ W(x,z,t) &= w_b(x,t) + w_s(x,t) \end{aligned} \tag{1}$$

where u is the axial displacement along the mid-plane of the beam, w_b and w_s are the bending and shear components of transverse displacement along the mid-plane of the beam, respectively. The non-zero strains are given by:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = \varepsilon_x^\circ + z\kappa_x^b + f\kappa_x^s \\ \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = (1-f')\gamma_{xz}^\circ = g\gamma_{xz}^\circ \end{aligned} \tag{2}$$

where

$$\begin{aligned} f &= z\left[-\frac{1}{4} + \frac{5}{3}\left(\frac{z}{h}\right)^2\right] \\ g &= 1 - f' = \frac{5}{4}\left[1 - 4\left(\frac{z}{h}\right)^2\right] \end{aligned} \tag{3}$$

and $\varepsilon_x^\circ, \gamma_{xz}^\circ, \kappa_x^b, \kappa_x^s$ and κ_{xy} are the axial strain, shear strains and curvatures in the beam, respectively defined as:

$$\begin{aligned} \varepsilon_x^\circ &= u' \\ \gamma_{xz}^\circ &= w_s' \\ \kappa_x^b &= -w_b'' \\ \kappa_x^s &= -w_s'' \end{aligned} \tag{4}$$

where differentiation with respect to the x-axis is denoted by primes (').

3. Variational Formulation

In order to derive the equations of motion, Hamilton's principle is used:

$$\delta\int_{t_1}^{t_2}(K - U - V)dt = 0 \tag{5}$$

where U is the variation of the strain energy, V is the variation of the potential energy, and K is the variation of the kinetic energy.

The variation of the strain energy can be stated as:

$$\begin{aligned} \delta U &= \int_v \left(\sigma_x\delta\varepsilon_x + \sigma_{xz}\delta\gamma_{xz}\right)dv = \\ &= \int_0^l \left(N_x\delta\varepsilon_z^\circ + M_x^b\delta\kappa_x^b + M_x^s\delta\kappa_x^s + Q_{xz}\delta\gamma_{xz}^\circ\right)dx \end{aligned} \tag{6}$$

where N_x, M_x^b, M_x^s and Q_{xz} are the axial force, bending moments and shear force, respectively, defined by integrating over the cross-sectional area A as:

$$\begin{aligned} N_x &= \int_A \sigma_x dA \\ M_x^b &= \int_A \sigma_x z dA \\ M_x^s &= \int_A \sigma_x f dA \\ Q_{xz} &= \int_A \sigma_{xz} g dA \end{aligned} \tag{7}$$

The variation of the potential energy of the axial force can be expressed as:

$$\delta V = -\int_0^l P_0\left[\delta w_{b'}\left(w_{b'} + w_{s'}\right) + \delta w_{s'}\left(w_{b'} + w_{s'}\right)\right]dx \tag{8}$$

The variation of the kinetic energy is obtained as:

$$\begin{aligned} \delta K &= \int_v \rho_k\left(\dot{U}\delta\dot{U} + \dot{V}\delta\dot{V} + \dot{W}\delta\dot{W}\right)dv = \\ &= \int_0^l [\delta\dot{u}\left(m_0\dot{u} - m_1\dot{w}_b' - m_f\dot{w}_s'\right) + \\ &+ \delta\dot{w}_b'\left(-m_1\dot{u} + m_2\dot{w}_b' + m_{fz}\dot{w}_s'\right) + \\ &+ \delta\dot{w}_s m_0\left(\dot{w}_b + \dot{w}_s\right) + \\ &+ \delta\dot{w}_s'\left(-m_f\dot{u} + m_{fz}\dot{w}_b' + m_{f^2}\dot{w}_s'\right)]dx \end{aligned} \tag{9}$$

where the differentiation with respect to the time t is denoted by dot-superscript convention and ρk is the density of a kth layer and m_0, m_1, m_2, m_f, m_{fz} and m_{f2} are the inertia coefficients, defined by:

$$\begin{aligned} m_f &= -\frac{m_1}{4} + \frac{5}{3h^2}m_3 \\ m_{fz} &= -\frac{m_2}{4} + \frac{5}{3h^2}m_4 \\ m_{f^2} &= \frac{m_2}{16} - \frac{5}{6h^2}m_4 + \frac{25}{9h^4}m_6 \end{aligned} \tag{10}$$

where

$$\left(m_0, m_1, m_2, m_3, m_4, m_6\right) = \int_A \rho_k\left(1, z, z^2, z^3, z^4, z^6\right)dA \tag{11}$$

By substituting Eqs. (6), (8) and (9) into Eq. (5), the following weak statement is obtained:

$$\begin{aligned} 0 = \int_{t_2}^{t_2}\int_0^l [&\delta\dot{u}\left(m_0\dot{u} - m_1\dot{w}_b' - m_f\dot{w}_s'\right) + \\ &+ \delta\dot{w}_b m_0\left(\dot{w}_b + \dot{w}_s\right) + \\ &+ \delta\dot{w}_b'\left(-m_1\dot{u} + m_2\dot{w}_b' + m_{fz}\dot{w}_s'\right) + \\ &+ \delta\dot{w}_s m_0\left(\dot{w}_b + \dot{w}_s\right) + \\ &+ \delta\dot{w}_s'\left(-m_f\dot{u} + m_{fz}\dot{w}_b' + m_{f2}\dot{w}_s'\right) + \\ &+ P_0\left[\delta w_{b'}\left(w_{b'} + w_{s'}\right) + \delta w_{s'}\left(w_{b'} + w_{s'}\right)\right] - \\ &+ N_x\delta u' + M_x^b\delta w_b'' + M_x^s\delta w_s'' - Q_{xz}\delta w_s']dxdt \end{aligned} \tag{12}$$

4. Constitutive equations

The stress-strain relations for the kth lamina are given by:

$$\begin{aligned} \sigma_x &= \bar{Q}_{11}\gamma_x \\ \sigma_{xz} &= \bar{Q}_{55}\gamma_{xz} \end{aligned} \tag{13}$$

where $\bar{Q}_{11}$ and $\bar{Q}_{55}$ are the elastic stiffnesses transformed to the x-direction. More detailed explanation can be found in (Jones, 1999).

The constitutive equations for bar forces and bar strains are obtained by using Eqs. (2), (7) and (13):

$$\begin{Bmatrix} N_x \\ M_x^b \\ M_x^s \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ & R_{22} & R_{23} & 0 \\ & & R_{33} & 0 \\ \text{sym.} & & & R_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^\circ \\ \kappa_x^b \\ \kappa_x^s \\ \gamma_{xz}^\circ \end{Bmatrix} \tag{14}$$

where R_{ij} are the laminate stiffnesses of general composite beams and given by:

$$\begin{aligned} R_{11} &= \int_y A_{11}dy \\ R_{12} &= \int_y B_{11}dy \\ R_{13} &= \int_y \left(-\frac{B_{11}}{4} + \frac{5}{3h^2}E_{11}\right)dy \\ R_{22} &= \int_y D_{11}dy \\ R_{23} &= \int_y \left(-\frac{D_{11}}{4} + \frac{5}{3h^2}F_{11}\right)dy \\ R_{33} &= \int_y \left(\frac{D_{11}}{16} - \frac{5}{6h^2}F_{11} + \frac{25}{9h^4}H_{11}\right)dy \\ R_{44} &= \int_y \left(\frac{25}{16}A_{55} - \frac{25}{2h^2}D_{55} + \frac{25}{h^4}F_{55}\right)dy \end{aligned} \tag{15}$$

where A_{ij}, B_{ij} and D_{ij} matrices are the extensional, coupling and bending stiffness and E_{ij}, F_{ij}, H_{ij} matrices are the higher-order stiffnesses, respectively, defined by:

$$\left(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}\right) = \int_z \bar{Q}_{ij}\left(1, z, z^2, z^3, z^4, z^6\right)dz \tag{16}$$

5. Governing equations of motion

The equilibrium equations of the present study can be obtained by integrating the derivatives of the varied quantities by parts and collecting the coefficients of δu, δu_b and δu_s:

$$\begin{aligned} & N_x' = m_0\ddot{u} - m_1\ddot{w}_b' - m_f\ddot{w}_s'; \\ & M_x^{b''} - P_0\left(w_{b''} + w_{s''}\right) = \\ & = m_0\left(\ddot{w}_b + \ddot{w}_s\right) + m_1\ddot{u}' - m_2\ddot{w}_b'' - m_{fz}\ddot{w}_s''; \\ & M_x^{s''} + Q_{xz}' - P_0\left(w_{b''} + w_{s''}\right) = \\ & = m_0\left(\ddot{w}_b + \ddot{w}_s\right) + m_f\ddot{u}' - m_{fz}\ddot{w}_b'' - m_{f2}\ddot{w}_s'' \end{aligned} \tag{17}$$

The natural boundary conditions are of the form:

$$\begin{aligned} & \delta u : N_x \\ & \delta w_b : M_x^{b'} - P_0\left(wb' + ws'\right) - m_1\ddot{u} + m_2\ddot{w}_b' + m_{fz}\ddot{w}_s' \\ & \delta w_b' : M_x^b \\ & \delta w_s : M_x^{s'} + Q_{xz} - P_0\left(wb' + ws'\right) - m_f\ddot{u} + m_{fz}\ddot{w}_b' + m_{f2}\ddot{w}_s' \\ & \delta w_s' : M_x^s \end{aligned} \tag{18}$$

By substituting Eqs. (4) and (14) into Eq. (17), the explicit form of the governing equations of motion can be expressed with respect to the laminate stiffnesses R_{ij}. Eq. (17) is the most general form for axial-flexural coupled vibration of axially loaded of composite beams, and the dependent variables, u, w_b and w_s are fully coupled.

6. Finite element formulation

The present theory for composite beams described in the previous section was implemented via a displacement based finite element method. The variational statement in Eq. (12) requires that the bending and shear components of transverse displacement w_b and w_s be twice differentiable and C^1-continuous, whereas the axial displacement u must be only once differentiable and C^0-continuous. The generalized displacements are expressed over each element as a combination of the linear interpolation function Ψ_j for u and Hermite-cubic interpolation function ψ_j for w_b and w_s associated with node j and the nodal values:

$$\begin{aligned} u &= \sum_{j=1}^{2} u_j\Psi_j \\ w_b &= \sum_{j=1}^{4} w_{bj}\psi_j \\ w_s &= \sum_{j=1}^{4} w_{sj}\psi_j \end{aligned} \tag{19}$$

Substituting these expressions in Eq. (19) into the corresponding weak statement in Eq. (12), the finite element model of a typical element can be expressed as the standard eigenvalue problem:

$$\left(\left[K\right] - P_0\left[G\right] - \omega^2\left[M\right]\right)\left\{\Delta\right\} = \left\{0\right\} \tag{20}$$

where $[K]$, $[G]$ and $[M]$ are the element stiffness matrix, the element geometric stiffness matrix and the element mass matrix, respectively. In Eq.(20), $\{\Delta\}$ is the eigenvector of nodal displacements corresponding to an eigenvalue:

$$\left\{\Delta\right\} = \left\{u\ w_b\ w_s\right\}^{\mathrm{T}} \tag{21}$$

7. Numerical examples

In this section, a number of numerical examples are presented and analysed for verification the accuracy of the present theory in predicting the natural frequencies, critical buckling loads and corresponding mode shapes of composite beams with arbitrary lay-ups. All laminate are of equal thickness and made of the same orthotropic material, whose properties are as follows:

Material I:

$$E_1 / E_2 = open,$$
$$G_{12} = G_{13} = 0.6E_2$$
$$G_{23} = 0.5E_2 \tag{22}$$
$$\nu_{12} = 0.25$$

Material II:

$$E_1 / E_2 = open$$
$$G_{12} = G_{13} = 0.5E_2$$
$$G_{23} = 0.2E_2 \tag{23}$$
$$\nu_{12} = 0.25$$

For convenience, the following non-dimensional terms are used in presenting the numerical results:

$$\overline{P_{cr}} = \frac{P_{cr}L^2}{E_2 bh^3}$$
$$\overline{\omega} = \frac{\omega L^2}{h}\sqrt{\rho E_2} \tag{24}$$

As the first example, vibration and buckling analysis of a symmetric and an anti-symmetric cross-ply composite beam with simply-supported boundary condition is performed. Material I and II with E_1/E_2 = 10 and 40 are used. The fundamental natural frequencies and critical buckling loads for different span-to-height *L/h* ratios are compared with exact solutions (Khdeir and Reddy, 1994; Khdeir and Reddy, 1997) and the finite elements results (Murthy et al., 2005; Aydogdu, 2005; Aydogdu, 2006a) in Tables 1 and 2. An excellent agreement between the predictions of the present model and the results of the other models mentioned can be observed.

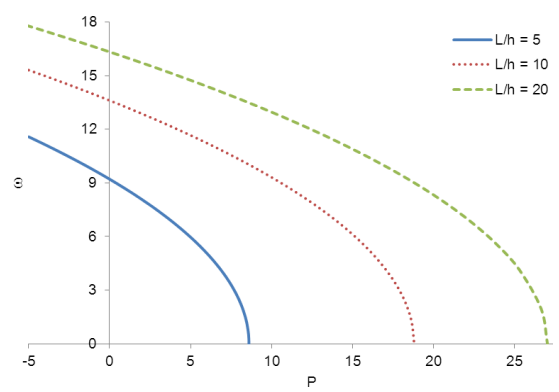
Material I with E_1/E_2 = 40 is chosen to show the effect of the axial force on the fundamental natural frequencies of beam with various *L/h* ratios (Fig. 2). It can be seen that the change of the natural frequency due to axial force is noticeable. The natural frequency diminishes when the axial force changes from tensile to compressive, as expected. It is obvious that the natural frequency decreases with the increase of axial force, and the decrease becomes more quickly when the axial force is close to critical buckling load. For an anti-symmetric cross-ply lay-up, with *L/h* = 5, 10 and 20, at about P = 3.903, 4.936 and 5.290, respectively, the natural frequencies become zero which implies that at these loads, bucklings occur as a degenerate case of natural vibration at zero frequency. It also means that the buckling loads of composite beams under axial force can be also obtained indirectly through vibration problem by increasing the axial force until the corresponding natural frequency vanishes. Besides, Fig. 2 explains the duality between the buckling load and natural frequency. In order to show the effect of material anisot-

Table 1.
Effect of span-to-height ratio on the fundamental natural frequencies of a simply-supported cross-ply composite beam (Material I with E_1/E_2 = 40).

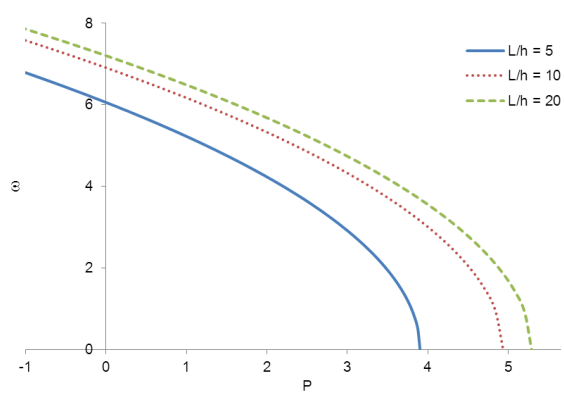
Lay-ups	Reference	*L/h*		
		5	10	20
$[0^0/90^0/0^0]$	Murthy et al. (2005)	9.207	13.614	-
	Khdeir and Reddy (1994)	9.208	13.614	-
	Aydogdu (2005)	9.207	-	16.337
	Present	9.206	13.607	16.327
$[0^0/90^0]$	Murthy et al. (2005)	6.045	6.908	-
	Khdeir and Reddy (1994)	6.128	6.945	-
	Aydogdu (2005)	6.144	-	7.218
	Present	6.058	6.909	7.204

Table 2.
Effect of span-to-height ratio on the critical buckling loads of a simply-supported cross-ply composite beam (Material I and II with E_1/E_2 = 40).

Lay-ups	Reference	*L/h*		
		5	10	20
Material I				
$[0^0/90^0/0^0]$	Khdeir and Reddy (1997)	8.613	18.832	-
	Aydogdu (2006a)	8.613	-	27.084
	Present	8.609	18.814	27.050
$[0^0/90^0]$	Aydogdu (2006a)	3.906	-	5.296
	Present	3.903	4.936	5.290
Material II				
$[0^0/90^0/0^0]$	Aydogdu (2006a)	5.896	-	24.685
	Present	5.895	14.857	24.655
$[0^0/90^0]$	Aydogdu (2006a)	3.376	-	5.225
	Present	3.373	4.697	5.219



a) Symmetric cross-ply lay-up ($[0^0/90^0/0^0]$)



b) Anti-symmetric cross-ply lay-up ($[0^0/90^0]$)

Figure 2. Load-frequency curves of a simply-supported cross-ply composite beam (*L/h* = 5, 10 and 20).

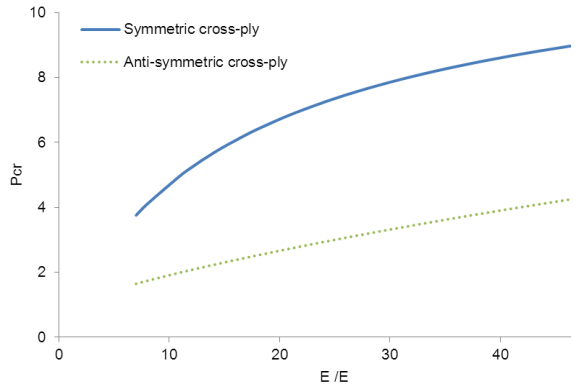
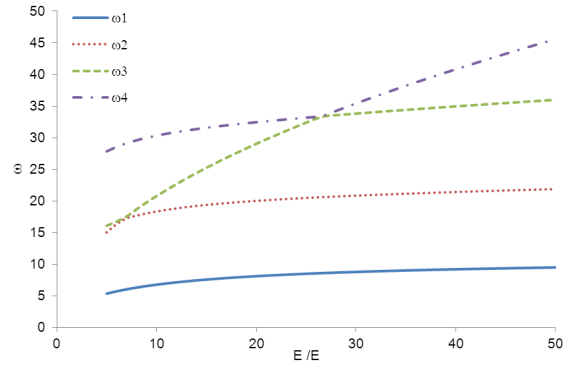


Figure 3. Effect of material anisotropy on the critical buckling loads of a simply-supported cross-ply composite beam (L/h = 5).

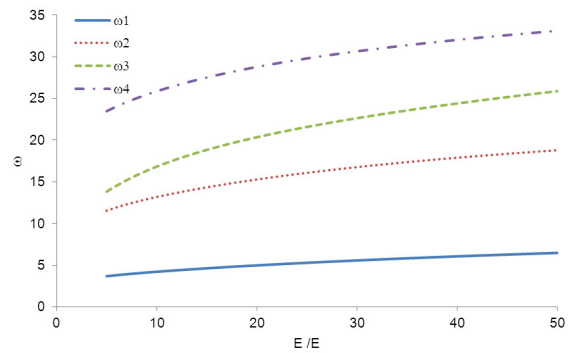
ropy (E_1/E_2) on the critical buckling loads and the first four natural frequencies of a symmetric and an anti-symmetric cross-ply lay-up, a simply-supported composite beam with L/h = 5 is performed. It is observed that the critical buckling loads and natural frequencies increase with increasing orthotropy (Figs. 3 and 4). For a symmetric cross-ply lay-up, as ratio of E_1/E_2 increases, the order of the second and third vibration mode as well as the third and fourth vibration mode changes each other at E_1/E_2 = 7 and 27, respectively (Fig. 4).

8. Conclusions

A two-noded C^1 beam element of five degree-of-freedom per node is developed to study the vibration and buckling behaviour of cross-ply composite beams using refined shear deformation theory. This model is capable of predicting accurately the natural frequencies, critical buckling loads and load-frequency interaction curves. It accounts for the parabolical variation of shear strains through the depth of the beam, and satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factor. The present model is found to be appropriate and efficient in analyzing vibration and buckling problem of cross-ply composite beams.



a) Symmetric cross-ply lay-up ([$0^0/90^0/0^0$])



b) Anti-symmetric cross-ply lay-up ([$0^0/90^0$])

Figure 4. Effect of material anisotropy on the first four natural frequencies of a simply-supported cross-ply composite beam (L/h = 5).

Reference

Aydogdu, M. (2005). Vibration analysis of cross-ply laminated beams with general boundary conditions by Ritz method. *International Journal of Mechanical Sciences*, Vol. 47, No 11, pp. 1740-1755.

Aydogdu, M. (2006a). Buckling analysis of cross-ply laminated beams with general boundary conditions by Ritz method. *Composites Science and Technology*, Vol. 66, No 10, pp. 1248-1255.

Aydogdu, M. (2006b). Free vibration analysis of angle-ply laminated beams with general boundary conditions. *Journal of Reinforced Plastics and Composites*, Vol. 25, No 15, pp. 1571-1583.

Chandrashekhara, K. and Bangera, K. (1992). Free vibration of composite beams using a refined shear flexible beam element. *Computers and Structures*, Vol. 43, No 4, pp. 719-727.

Chandrashekhara, K., Krishnamurthy, K. and Roy, S. (1990). Free vibration of composite beams including rotary inertia and shear deformation. *Composite Structures*, Vol. 14, No 4, pp. 269-279.

Chen, W.Q., Lv, C.F. and Bian, Z.G. (2004). Free vibration analysis of generally laminated beams via state-space-based differential quadrature. Composite Structures, Vol. 63, No 3-4, pp. 417-425.

Jones, R.M. (1999). *Mechanics of Composite Materials*, Philadelphia: Taylor and Francis.

Jun, L. and Hongxing, H. (2011). Free vibration analyses of axially loaded laminated composite beams based on higher-order shear deformation theory. *Meccanica*, Vol. 46, pp. 1299-1317.

Jun, L., Xiaobin, L. and Hongxing, H. (2009). Free vibration analysis of third-order shear deformable composite beams using dynamic stiffness method. *Archive of Applied Mechanics*, Vol. 79, pp. 1083-1098.

Kant, T., Marur, S.R. and Rao, G. (1997). Analytical solution to the dynamic analysis of laminated beams using higher order refined theory. *Composite Structures*, Vol. 40, No 1, pp. 1-9.

Kant, T. and Swaminathan, K. (2001). Analytical solutions for free vibration of laminated composite and sandwich plates based on a higher-order refined theory. *Composite Structures*, Vol. 53, pp. 73-85.

Khdeir, A.A. and Reddy, J.N. (1994). Free vibration of cross-ply laminated beams with arbitrary boundary conditions. *International Journal of Engineering Science*, Vol. 32, No. 12, pp. 1971-1980.

Khdeir, A.A. and Reddy, J.N. (1997). Buckling of cross-ply laminated beams with arbitrary boundary conditions. *Composite Structures*, Vol. 37, No 1, pp. 1-3.

Krishnaswamy, S., Chandrashekhara, K. and Wu, W.Z.B. (1992). Analytical solutions to vibration of generally layered composite beams. Journal of Sound and Vibration, Vol. 159, No 1, pp. 85-99.

Marur, S.R. and Kant, T. (1996). Free vibration analysis of fiber reinforced composite beams using higher order theories and finite element modelling. *Journal of Sound and Vibration*, Vol. 194, No 3, pp. 337-351.

Matsunaga, H. (2001). Vibration and buckling of multilayered composite beams according to higher order deformation theories. *Journal of Sound and Vibration*, Vol. 246, No 1, pp. 47-62.

Murthy, M.V.V.S., Mahapatra, D.R., Badarinarayana, K. and Gopalakrishnan, S. (2005). A refined higher order finite element for asymmetric composite beams. *Composite Structures*, Vol. 67, pp. 27-35.

Shi, G. and Lam, K.Y. (1999). Finite element vibration analysis of composite beams based on higher-order beam theory. *Journal of Sound and Vibration*, Vol. 219, No 4, pp. 707-721.

Subramanian, P. (2006). Dynamic analysis of laminated composite beams using higher order theories and finite elements. *Composite Structures*, Vol. 73, No 3, pp. 342-353.

Vo T.P. and Thai, H.T. (2012). Static behaviour of composite beams using various refined shear deformation theories. *Composite Structures*, in press.

Zhen, W. and Wanji, C. (2008). An assessment of several displacement-based theories for the vibration and stability analysis of laminated composite and sandwich beams. *Composite Structures*, Vol. 84, No 4, pp. 337-349.